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ON REID'S 3-SIMPLICIAL MATROID THEOREM

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In this paper we prove the following result of Ralph Reid (which was never published nor completely proved).

Theorem. Let M be a matroid coordinatizable (representable) over a prime field F. Then there is a 3-simplicial matroid M' over F which is a series extension of M.

The proof we give is different from the original proof of Reid which uses techniques of algebraic topology. Our proof is constructive and uses elementary matrix operations.

1.

All matroids considered in the present paper are on finite sets. Basic notions on matroids may be found in [6].

The definition of *simplicial matroids over the rationals,* which was introduced by Crapo and Rota [3, 4], generalizes to an arbitrary field [1, 2].

Let $A=\{a_0<a_1<\dots<a_{n-1}\}$ be a finite ordered set. For every $k, 0\leqq k\leqq n$, let $\binom{A}{k}$ denote the *set of all k-element subsets of A*. If $X\in\binom{A}{k}$ we may view X as a *simplex*. Also we may consider

(i) $\partial(\emptyset)=0$

(ii) $\partial(\{a_{i_0}\})=1$

(iii) $\partial(\{a_{i_0}<\dots<a_{i_{k-1}}\})=\sum\limits_{0}^{k-1}(-1)^j\{a_{i_0},\dots,a_{i_{j-1}},a_{i_{j+1}},\dots,a_{i_{k-1}}\}$

(the "boundary" of a simplex) as a vector over a (commutative) field F.

Definition 1.1 [1]. *A k-simplicial matroid of order n over a field F on a subset E of* $\binom{A}{k}$ is the matroid given by the coordinatization (representation) $e\to\partial(e)$, $e\in E$, i.e. $\{e_1, e_2, \dots, e_n\}\subseteqq E$ is an independent set of the matroid if and only if $\{\partial(e_1), \dots$

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$\ldots, \partial(e_n)\}$ is a family of linearly independent vectors. We denote by $S_k^n[F]$ the *full k-simplicial matroid of order n, over a field F,* on the set $\binom{A}{k}$ of all k-subsets of a set A of cardinality n.

Definition 1.2. Let $A=\{a_0<a_1<\ldots<a_{n-1}\}$ be a finite ordered set. For every k, $1\leqq k\leqq n$, we assume that the set $\binom{A}{k}$ is ordered by the lexicographic order. By definition a *simplicial matrix* is the matrix $S(n,k)=(s_{p,q})$ whose rows (resp. columns) are indexed by the set $\binom{A}{k-1}$ $\left(\text{resp. } \binom{A}{k}\right)$ and the coefficient $s_{p,q}$ is zero if $p\not\subset q$ and equals $(-1)^j$ if $p=q-\{a_{i_j}\}$, $q=\{a_{i_0}<\ldots<a_{i_j}<\ldots<a_{i_{k-1}}\}$.

The simplicial matrix $S(n,k)$ with entries in F is a *representative matrix* of the full k-simplicial matroid $S_k^n[F]$, i.e. $\{x_1,\ldots,x_m\}$ is an independent set of the matroid $S_k^n[F]$ if and only if the column vectors of the matrix $S(n,k)$ indexed by $x_1,\ldots,x_m$ are lineary independent.

Definition 1.3. Let $M(E)$ be a matroid on the set E, let $x\in E$ and suppose $y\notin E$. *The series extension of M at x by y* is the matroid M' on $E\cup\{y\}$ which has as its bases the sets of the form (1) or (2):

(1) $B\cup\{y\}$, B is a base of M,

(2) $B\cup\{x\}$, B is a base of M and $x\notin B$.

We define a *series extension* of a matroid M to be a matroid M' which can be obtained from M by successive series extensions. We say that the elements $e_1, e_2, \ldots, e_n$ of the matroid $M(E)$ are *in series* if they are coloops, or for all i,j, $1\leqq i<j\leqq n$, $\{e_i, e_j\}$ is a cocircuit.

Remark 1.4. Let $E\subset E'$ be two finite sets. The matroid $M(E')$ is a series extension of $M(E)$ if and only if $M(E)=M(E')/(E'-E)$ and there is a surjective function f from E' to E such that the restriction of f to E is the identity map of E and for every e, such that $f^{-1}(\{e\})=\{e': e'\in E' \text{ and } f(e')=e\}$ is not a singleton, the elements of the set $f^{-1}(\{e\})$ are in series on $M(E')$.

Let N be a matrix over the field F whose columns are indexed by the set E. Let $M(E)$ be the *matroid induced on the columns of the matrix N by linear independence* (i.e. such that N is its representative matrix). Let $e_1,\ldots,e_n$ be elements in series on the matroid $M(E)$ and let $C_1=\{e_1,\ldots,e_n,x_1,\ldots,x_m\}$, $C_2=\{e_1,\ldots,e_n,y_1,\ldots,y_{m'}\}$, be two circuits of $M(E)$. If we have

$$\sum_1^n \alpha_i\bar{e}_i+\sum_1^m \beta_i\bar{x}_i=0 \quad\text{and}\quad \sum_1^n \alpha_i'\bar{e}_i+\sum_1^{m'}\beta_i'\bar{y}_i=0$$

where $\alpha_i, \alpha_i', \beta_i, \beta_i'\in F-\{0\}$ then necessarily $\sum\limits_1^n \alpha_i\bar{e}_i=\gamma\cdot\left(\sum\limits_1^n \alpha_i'\bar{e}_i\right)$ where $\gamma\in F-\{0\}$. Then, for every e_i, $1\leqq i\leqq n$, the matroid $M'=M/\{e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n\}$ is the matroid of linear dependence of the columns of the matrix N' obtained from the matrix N replacing the set of column vectors $\{\bar{e}_1,\ldots,\bar{e}_n\}$ by the column vector $\bar{e}=\gamma'\cdot\sum\limits_1^n \alpha_i\bar{e}_i$, where $\gamma'\in F-\{0\}$. If $e_1,\ldots,e_n$ are coloops of the matroid M, then clearly $M/\{e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n\}=M-\{e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n\}$.

Example 1.5. Let N be the matrix below over the field F whose columns are indexed by the set $\{e_1, \ldots, e_5\}$:

$$N = \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Let $M(\{e_1, \ldots, e_5\})$ be the matroid of linear dependence of the columns of the matrix N. The elements e_2 and e_4 are in series, because the unique circuits which contain e_2 (resp. e_4) are $\{e_1, e_2, e_4\}$ and $\{e_2, e_3, e_4, e_5\}$. The identities $\bar{e}_1+\bar{e}_2+\bar{e}_4=0$ and $\bar{e}_2+\bar{e}_3+\bar{e}_4+\bar{e}_5=0$ imply that the isomorphic matroids $M/\{e_2\}$ and $M/\{e_4\}$ have the matrix N' below as their representative matrix over the field F:

$$N' = \begin{bmatrix} 1 & -\alpha & 1 & 0 \\ -1 & \alpha & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

where $\alpha \in F - \{0\}$.

2.

Theorem 2.1. [5]. *Let M be a matroid coordinatizable over a prime field F, $F=\mathbf{Q}$ or $F=\mathbf{Z}_p$. Then there is a 3-simplicial matroid M' over F which is a series extension of M.*

Lemma 2.2. *Let q be a positive integer and let n be the integer $[3q/2]+3$. Then there are triangles $T_1, T_2, \ldots, T_m \in \left\{\binom{\{1, 2, \ldots, n\}}{3} - \{1, 2, 3\}\right\}$ and scalars α_i, $1 \leqq i \leqq m$, $\alpha_i = 1$ or $\alpha_i = -1$, such that $\Sigma \alpha_i \cdot \partial(T_i) = q \cdot \partial(\{1, 2, 3\})$.*

Proof of Lemma 2.2. Let $T_1, \ldots, T_m$ be the triangles of one of Figures 1 or 2. For all i, $1 \leqq i \leqq m$, let α_i be equal to 1 or -1 if the triangle T_i is affected respectively by the signs $+$ or $-$. The definition of ∂ implies that $\Sigma \alpha_i \cdot \partial(T_i) = q \cdot \partial(\{1, 2, 3\})$. Since we can generalize the construction of Figure 1 (resp. Figure 2) for all even integers (resp. odd integers) the lemma is proved. ∎

Proof of Theorem 2.1. Let M be a matroid coordinatizable over a prime field F. Without loss of generality we suppose that M has no loops. For sufficently large n we shall prove the existence of a submatrix N' of $S(n, 3)$ (with entries in F) composed by some of its columns and such that the matroid M', induced by linear independence on the N' columns, is a series extension of M.

It is clear from basic linear algebra that there exists a representative matrix M (with entries in F) of the form $[I{:}N]$ where I is the $r \times r$ unity matrix and each entry a_{ij} of the matrix N is a integer.

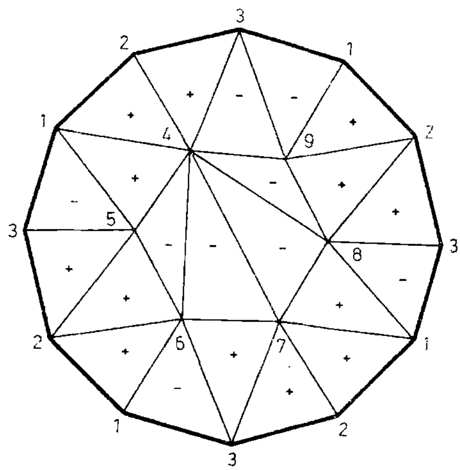

Fig. 1

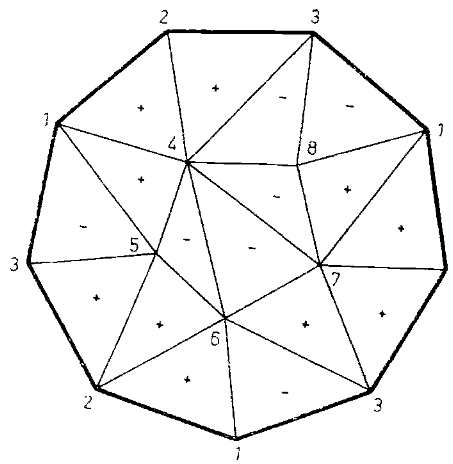

Fig. 2

With every non-zero entry a_{ij} of the matrix N, we attach a family of triangles Δ_{ij} determined by the following three conditions* (the construction is possible by Lemma 2.2):

(I) If v is an integer labelling a vertex of a triangle of the family Δ_{ij} and if $v\neq 1, 2, i+2$ then $v\geqq r+3$.

(II) Let $\Delta_{ij}=\{T_1, \ldots, T_m\}$. Then $T_s\neq\{1, 2, i+2\}$, $1\leqq s\leqq m$, and we have $\sum_1^m \alpha_s\cdot\partial(T_s)=|a_{ij}|\cdot\partial(\{1, 2, i+2\})$, where the scalars α_s are equal to 1 or -1 (in particular, $\alpha_s=-1$ if $1, i+2\in T_s$) and $|a_{ij}|$ is equal to the absolute value of a_{ij} if $F=\mathbf{Q}$ or equal to a_{ij} if $F\neq\mathbf{Q}$.

(III) If T and T' are two triangles such that $T\in\Delta_{ij}$, $T'\in\Delta_{i'j'}$ and $i\neq i'$ then $T\cap T'\subset\{1, 2\}$.

We construct the matrix N' (a submatrix of the simplicial matrix $S(n, 3)$) as follows (Conditions 1—6):

(1) The first r columns of the matrix N' are the columns of the matrix $S(n, 3)$ indexed by the triangles $\{1, 2, 3\}, \ldots, \{1, 2, r+2\}$.

Let $a_{i_1 j}, \ldots, a_{i_h j}$ be the non-zero entries of the column j of the matrix N. *The column j of N creates the following columns in the matrix N'* (Condition 2 refers to the case $h=1$ and Conditions 3—6 refer to the case $h\geqq 2$):

(2) If $h=1$ (i.e. if there exists only one non-zero entry in the column j of N) then the columns of $S(n, 3)$ indexed by the triangles $\Delta_{i_1 j}$ are columns of the submatrix N'.

Suppose that $h\geqq 2$.

(3) For every family Δ_{ij} of triangles let $\{1, 2, \alpha_{ij}\}$ be its first triangle (in the lexico-

* The vertices of the triangles Δ_{ij} are labelled by integers. We identify a triangle to the set of the integers labelling its vertices.

graphic order of its vertices). Then the columns of $S(n, 3)$ indexed by the triangles $\bigcup_{s=i_1}^{i_h} (\Delta_{sj} - \{1, 2, \alpha_{sj}\})$ are columns of the submatrix N'.
(4) Let v be the greatest integer labelling a vertex of a triangle of $\bigcup_{i,j} \Delta_{ij}$. Then the columns of $S(n, 3)$ indexed by the $3h-2$ triangles appearing in Figure 3 where $j_s = v + (j-1)\cdot 2r + s$, $1 \leqq s \leqq 2h$, are columns of the matrix N'. (Note that the h triangles $\{1, j_{2s-1}, j_{2s}\}$, $1 \leqq s \leqq h$, are removed in Figure 3. ($h=3$))

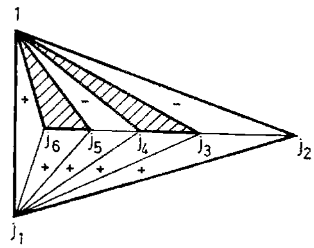

Fig. 3

Suppose that $F = \mathbf{Q}$.
(5) For every positive (resp. negative) entry $a_{i_s j}$, $1 \leqq s \leqq h$, of the column j of N, the columns of $S(n, 3)$ indexed by the four triangles appearing in Figure 4 (resp. Figure 5) are columns of N'.
Suppose that $F \neq \mathbf{Q}$.
(6) For every entry $a_{i_s j}$, $1 \leqq s \leqq h$, of the column j of N, the columns of $S(n, 3)$ indexed by the four triangles appearing in Figure 4 are columns of N'.

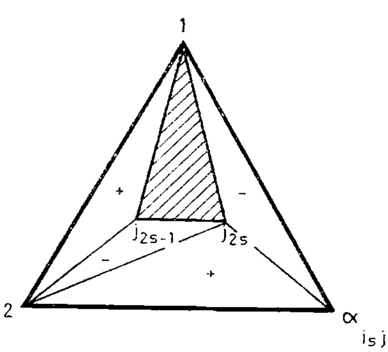

Fig. 4

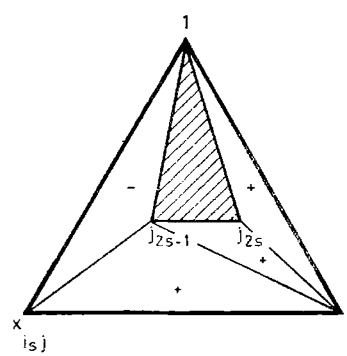

Fig. 5

We remark that *the columns* $T_1, \ldots, T_{m'}$, *of the matrix* $S(n, 3)$ *associated by Conditions* 2—6 *to the column* j *of* N, *are (by construction) in series in the matroid* M' *induced on the* N' *columns by linear independence.*

Let $a_{i_1 j}, \ldots, a_{i_h j}$ be again the non-zero entries of the column j of N. We claim that the column vectors of N' indexed by the triangles $\bigcup_{s=1}^{h} \{1, 2, i_s + 2\} \cup T_1 \cup \ldots \cup T_{m'}$, constitute a circuit of the matroid M'. To see this we note that Conditions 2—6 imply that there are scalars α_i equal to 1 or -1 such that

$$\text{(a)} \qquad \sum_{i=1}^{m'} \alpha_i \cdot \partial(T_i) = \sum_{s=1}^{h} a_{i_s j} \cdot \partial(\{1, 2, i_s+2\}).$$

The submatrix of $S(n,3)$ composed by its $\binom{n-1}{2}$ last rows is of the form $[I \vdots S(n-1,3)]$, where $S(n-1,3)$ is also a simplicial matrix. Since the rank of the full 3-simplicial matroid $S_3^n[F]$ is $\binom{n-1}{2}$, this matrix is also a representative matrix of the matroid $S_3^n[F]$ [2]. Hence the submatrix N'' of N' composed by its $\binom{n-1}{2}$ last rows is also a representative matrix of the matroid M'. We note that N'' is of the form $\left(\begin{array}{c:c} I & A \\ \hdashline O & B \end{array}\right)$ where I is the $r \times r$ unity matrix. Let $\overline{T}_1, \ldots, \overline{T}_{m'}$, be the column vectors of N'' attached to the column vector j of N. Then by the equality (a) above, the vector $\sum_{i=1}^{m'} \alpha_i \overline{T}_i$ (where the α_i's are the scalars determined by the equality (a)) has its first r entries equal to the corresponding entries of the column vector j of N and the remaining entries equal to zero. This implies, by Remark 1.4, that the matroid M' is a series extension of M. ∎

3.

Let $\mathcal{U}_{3,4}$ be the uniform matroid of rank 3 on 4 elements. It is clear that the following matrix A, over the field Q, is a representative matrix of the matroid $\mathcal{U}_{3,4}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We will illustrate Theorem 2.1 by constructing a submatrix N' of $S(16,3)$ composed by some of its columns and such that the matroid M', induced on the N' columns by linear independence, is a series extension of $\mathcal{U}_{3,4}$. (We remark that $\mathcal{U}_{3,4}$ is a 3-simplicial matroid because $S(4,3)$ is also a representative matrix of the matroid $\mathcal{U}_{3,4}$.) Triangles like $\{1,2,3\}$ will be abbreviated like 123.

By definition the matrix N' is the submatrix of $S(16,3)$ composed of the following columns:

(i) By Condition 1, the columns indexed by the triangles 123, 124, 125;

(ii) By Condition 3, the columns indexed by the triangles

$$\Delta_{11} - \{1,2,6\} \cup \Delta_{21} - \{1,2,7\} \cup \Delta_{31} - \{1,2,10\} \;,$$

where (by Conditions I—III)

$\Delta_{11} = \{126, 136, 236\}$

$\Delta_{21} = \{127, 128, 147, 149, 189, 248, 249, 279, 478, 789\}$

$\Delta_{31} = \{\{1,2,10\}, \{1,5,10\}, \{2,5,10\}\}$;

(iii) By Condition 4, the columns indexed by the triangles {1, 11, 16}, {1, 12, 13}, {1, 14, 15}, {11, 12, 13}, {11, 13, 14}, {11, 14, 15}, {11, 15, 16};
(iv) By Condition 5, the columns indexed by the triangles
{1, 2, 11}, {1, 6, 12}, {2, 6, 12}, {2, 11, 12},
{1, 2, 13}, {1, 7, 14}, {2, 7, 14}, {2, 13, 14} and
{1, 2, 16}, {1, 10, 15}, {2, 10, 15}, {2, 15, 16}.

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